NON-COMMUTATIVE CHERN CHARACTERS OF THE C*-ALGEBRAS OF SPHERES AND QUANTUM SPHERES

Nguyen Quoc Tho

Department of Mathematics, Vinh University, Vietnam

Abstract. We propose in this paper the construction of non-commutative Chern characters of the C^*-algebras of spheres and quantum spheres. The final computation gives us a clear relation with the ordinary $\mathbf{Z}/(2)$-graded Chern characters of tori or their normalizers.

Introduction.

For compact Lie groups the Chern character $ch : K^*(G)\otimes\mathbb{Q} \longrightarrow H^*_{DR}(G;\mathbb{Q})$ were constructed. In [4] - [5] we computed the non-commutative Chern characters of compact Lie group C^*-algebras and of compact quantum groups, which are also homomorphisms from quantum K-groups into entire current periodic cyclic homology of group C^*-algebras (resp., of C^*-algebra quantum groups), $ch_{C^*} : K_*(C^*(G)) \longrightarrow HE_*(C^*(G))$, (resp., $ch_{C^*} : K_*(C^*_\varepsilon(G)) \longrightarrow HE_*(C^*_\varepsilon(G)))$. We obtained also the corresponding algebraic vesion $ch_{\mathrm{alg}} : K_*(C^*(G)) \longrightarrow HP_*(C^*(G))$, which coincides with the Fedosov-Cuntz-Quillen formula for Chern characters [5]. When $A = C^*_\varepsilon(G)$ we first computed the K-groups of $C^*_\varepsilon(G)$ and the $HE_*(C^*_\varepsilon(G))$. Thereafter we computed the Chern chacracter $ch_{C^*} : K_*(C^*_\varepsilon(G)) \longrightarrow HE_*(C^*_\varepsilon(G)$ as an isomorphism modulo torsions.

Using the results from [4] - [5], in this paper we compute the non-commutative Chern characters $ch_{C^*} : K_*(A) \longrightarrow HE_*(A)$, for two cases $A = C^*(S^n)$, the C^*-algebra of spheres and $A = C^*_\varepsilon(S^n)$, the C^*-algebras of quantum spheres. For compact groups $G = O(n+1)$, the Chern character $ch : K_*(S^n)\otimes\mathbb{Q} \longrightarrow H^*_{DR}(S^n;\mathbb{Q})$ of the sphere $S^n = O(n+1)/O(n)$ is an isomorphism (see, [15]). In the paper, we describe two Chern character homomorphisms

$$ch_{C^*} : K_*(C^*(S^n)) \longrightarrow HE_*(C^*(S^n))$$

and

$$ch_{C^*} : K_*(C^*_\varepsilon(S^n)) \longrightarrow HE_*(C^*_\varepsilon(S^n)).$$

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Finally, we show that there is a commutative diagram

$$\begin{array}{ccc}
K_*\big(C^*(S^n)\big) & \xrightarrow{ch_{C^*}} & HE_*\big(C^*(S^n)\big) \\
\Big\downarrow \cong & & \Big\downarrow \cong \\
K_*\big(\mathbb{C}(\mathcal{N}_{\mathbf{T}_n})\big) & \xrightarrow{ch_{CQ}} & HE_*\big(\mathbb{C}(\mathcal{N}_{\mathbf{T}_n})\big) \\
\Big\downarrow \cong & & \Big\downarrow \cong \\
K^*\big(\mathcal{N}_{\mathbf{T}_n}\big) & \xrightarrow{ch} & H^*_{DR}\big(\mathcal{N}_{\mathbf{T}_n}\big)
\end{array}$$

(Similarly, for $A = C^*_\varepsilon(S^n)$, we have an analogous commutative diagram with $W \times S^1$ of place of $W \times S^n$), from which we deduce that ch_{C^*} is an isomorphism modulo torsions.

We now briefly review the structure of the paper. In section 1, we compute the Chern chatacter of the C^*-algebras of spheres. The computation of Chern character of $C^*(S^n)$ is based in two crucial points:

i) Because the sphere $S^n = O(n+1)/O(n)$ is a homogeneous space and C^*-algebra of S^n is the transformation group C^*-algebra, following J. Parker [10], we have, $C^*(S^n) \cong C^*(O(n)) \otimes \mathcal{K}(L^2(S^n))$.

ii) Using the stability property theorem K_* and HE_* in [5], we reduce it to the computation of C^*-algebras of subgroup $O(n)$ in $O(n+1)$ group.

In section 2, we compute the Chern character of C^*-algebras of quantum spheres. For quantum sphere S^n, we define the compact quantum C^*-algebra $C^*_\varepsilon(S^n)$, where ε is a positive real number. Thereafter, we prove that

$$C^*_\varepsilon(S^n) \cong \mathbb{C}(S^1) \oplus \bigoplus_{e \neq \omega \in W} \int_{S^1}^{\oplus} \mathcal{K}(H_{w,t})dt,$$

where $\mathcal{K}(H_{w,t})$ is the elementary algebra of compact operators in a separable infinite dimensional Hilbert space $H_{w,t}$ and W is the Weyl of a maximal torus $\mathbf{T}_n$ in $SO(n)$.

Similar to Section 1, we first compute the $K_*(C^*_\varepsilon(S^n))$ and $HE_*(C^*_\varepsilon(S^n))$, and we prove that $ch_{C^*} : K_*(C^*_\varepsilon(S^n)) \longrightarrow HE_*(C^*_\varepsilon(S^n))$ is a isomorphism modulo torsions.
Notes on Notation: For any compact space X, we write $K^*(X)$ for the $\mathbf{Z}/(2)$-graded topological K-theory of X. We use Swan's theorem to identify $K^*(X)$ with $\mathbf{Z}/(2)$-graded $K_*(\mathbb{C}(X))$. For any involutive Banach algebra $A, K_*(A)$, $HE_*(A)$, $HP_*(A)$ are $\mathbf{Z}/(2)$- graded algebraic or topological K-groups of A, entire cyclic homology, and periodic cyclic homology of A, respectively. If $\mathbf{T}$ is a maximal torus of a compact group G, with the corresponding Weyl group W, write $\mathbb{C}(\mathbf{T})$ for the algebra of complex valued functions on $\mathbf{T}$. We use the standard notations from the root theory such as P, P^+ for the positive highest weights, etc... We denote by $\mathcal{N}_{\mathbf{T}}$ the normalizer of $\mathbf{T}$ in G, by $\mathbb{N}$ the set of natural numbers, $\mathbb{R}$ the fied of real numbers and $\mathbb{C}$ the field of complex numbers, $\ell^2_A(\mathbb{N})$ the standard ℓ^2 space of square integrable sequences of elements from A, and finally by $C^*_\varepsilon(G)$ we denote the compact quantum algebras, $C^*(G)$ the C^*-algebra of G.

§1. Non-commutative Chern characters of C*-algebras of spheres.

In this section, we compute non-commutative Chern characters of C^*-algebras of spheres. Let A be an involutive Banach algebra. We construct the non-commutative Chern characters $ch_{C^*} : K_*(A) \longrightarrow HE_*(A)$, and show in [4] that for C^*-algebra $C^*(G)$ of compact Lie groups G, the Chern character ch_{C^*} is an isomorphism.

Proposition 1.1 ([5], Theorem 2.6). *Let H be a separable Hilbert space and B an arbitrary Banach space. We have*

i)
$$K_*(\mathcal{K}(H)) \cong K_*(\mathbb{C})$$
$$K_*(B \otimes \mathcal{K}(H)) \cong K_*(B);$$

ii)
$$HE_*(\mathcal{K}(H)) \cong HE_*(\mathbb{C})$$
$$HE_*(B \otimes \mathcal{K}(H)) \cong HE_*(B),$$

where $\mathcal{K}(H)$ is the elementary algebra of compact operators in a separable infitite-dimensional Hilbert space H. □

Proposition 1.2. ([5], Theorem 3.1). *Let A be an involutive Banach algebra with unity. There is a Chern character homomorphism*

$$ch_{C^*} : K_*(A) \longrightarrow HE_*(A).$$

□

Proposition 1.3. ([5], Theorem 3.2). *Let G be an compact group and $\mathbf{T}$ a fixed maximal torus of G with Weyl group $W := \mathcal{N}_{\mathbf{T}}/\mathbf{T}$. Then the Chern character $ch_{C^*} : K_*(C^*(G)) \longrightarrow HE_*(C^*(G))$ is an isomorphism modulo torsions, i.e.*

$$ch_{C^*} : K_*(C^*(G)) \otimes \mathbb{C} \xrightarrow{\cong} HE_*(C^*(G)),$$

which can be identified with the classical Chern character

$$ch : K_*(C(\mathcal{N}_{\mathbf{T}})) \longrightarrow HE_*(C(\mathcal{N}_{\mathbf{T}})),$$

that is also an isomorphism modulo torsions, i.e.

$$ch : K^*(\mathcal{N}_{\mathbf{T}}) \otimes \mathbb{C} \xrightarrow{\cong} H^*_{DR}(\mathcal{N}_{\mathbf{T}}).$$

□

Now, for $S^n = O(n+1)/O(n)$, where $O(n)$, $O(n+1)$ are the orthogonal matrix groups. We denote by $\mathbf{T}_n$ a fixed maximal torus of $O(n)$ and $\mathcal{N}_{\mathbf{T}_n}$ the normalizer of $\mathbf{T}_n$ in $O(n)$. Following Proposition 1.2, there a natural Chern character $ch_{C^*} : K_*(C^*(S^n)) \longrightarrow HE_*(C^*(S^n))$. Now, we compute first $K_*(C^*(S^n))$ and then $HE_*(C^*(S^n))$ of C^*-algebra of the sphere S^n.

Proposition 1.4.

$$HE_*(C^*(S^n)) \cong H^W_{DR}(\mathbf{T}_n)$$

Proof: We have

$$\begin{aligned} HE_*(C^*(S^n)) &= HE_*\big(C^*(O(n+1)/O(n))\big) \\ &\cong HE_*(C^*(O(n)) \otimes \mathcal{K}\big(L^2\big(O(n+1)/O(n)\big)\big)\big) \end{aligned}$$

(in virtue of [10], the $\mathcal{K}\big(L^2\big(O(n+1)/O(n)\big)\big)$ is a C^*-algebra compact operators in a separable Hilbert space $L^2\big(O(n+1)/O(n)\big)\big)$

$$\begin{aligned}
&\cong HE_*(C^*(O(n))) && \text{(by Proposition 1.1)}\\
&\cong HE_*(\mathbb{C}(\mathcal{N}_{\mathbf{T}_n})) && \text{(see [5])}.
\end{aligned}$$

Thus, we have $HE_*(C^*(S^n)) \cong HE_*(\mathbb{C}(\mathcal{N}_{\mathbf{T}_n}))$.

Apart from that, because $\mathbb{C}\left(\mathcal{N}_{\mathbf{T}_n}\right)$ is the commutative C^*-algebra, by a result Cuntz-Quillen's [1], we have an isomorphism

$$HP_*(\mathbb{C}(\mathcal{N}_{\mathbf{T}_n})) \cong H^*_{DR}(\mathcal{N}_{\mathbf{T}_n}).$$

Moreover, by a result of Khalkhali [8] - [9], we have

$$HP_*(\mathbb{C}(\mathcal{N}_{\mathbf{T}_n})) \cong HE_*(\mathbb{C}(\mathcal{N}_{\mathbf{T}_n})).$$

We have, hence

$$\begin{aligned}
HE_*(C^*(S^n)) &\cong HE_*(\mathbb{C}(\mathcal{N}_{\mathbf{T}_n})) \cong HP_*(\mathbb{C}(\mathcal{N}_{\mathbf{T}_n}))\\
&\cong H^*_{DR}(\mathcal{N}_{\mathbf{T}_n}) \cong H^W_{DR}(\mathbf{T}_n) \qquad \text{(by [15])}.
\end{aligned}$$

□

Remark 1. Because $H^W_{DR}(\mathbf{T}_n)$ is the de Rham cohomology of $\mathbf{T}_n$, invariant under the action of the Weyl group W, following Watanabe [15], we have a canonical isomorphism $H^W_{DR}(\mathbf{T}_n) \cong H^*(SO(n)) = \Lambda_{\mathbf{C}}(x_3, x_7, ..., x_{2i+3})$, where $x_{2i+3} = \sigma^*(p_i) \in H^{2n+3}(SO(n))$ and $\sigma^* : H^*(BSO(n), R), \longrightarrow H^*(SO(n), R)$ for a commutative ring R with a unit $1 \in R$, and $p_i = \sigma_i(t_1^2, t_2^2, ..., t_i^2) \in H^*(B\mathbf{T}_n, \mathbf{Z})$ the Pontryagin classes.

Thus, we have

$$HE_*(C^*(S^n)) \cong \Lambda_{\mathbf{C}}(x_3, x_7, ..., x_{2i+3}).$$

Proposition 1.5.

$$K_*(C^*(S^n)) \cong K^*(\mathcal{N}_{\mathbf{T}_n}).$$

Proof. We have

$$\begin{aligned}
K_*\big(C^*(S^n)\big) &= K_*\big(C^*(O(n+1)/O(n))\big)\\
&\cong K_*\big(C^*(O(n)) \otimes \mathcal{K}\big(L^2(O(n+1)/O(n))\big)\big) && \text{(see [10])}\\
&\cong K_*\big(C^*(O(n))\big) && \text{(by Proposition 1.1)}\\
&\cong K_*\big(\mathbb{C}(\mathcal{N}_{\mathbf{T}_n})\big)\\
&\cong K^*\big(\mathcal{N}_{\mathbf{T}_n}\big) && \text{(by Lemma 3.3, from [5])}.
\end{aligned}$$

Thus, $\qquad K_*(C^*(S^n)) \cong K^*(\mathcal{N}_{\mathbf{T}_n}).$ □

Remark 2. Following Lemma 4.2 from [5], we have

$$\begin{aligned}
K^*(\mathcal{N}_{\mathbf{T}_n}) &\cong K^*\big(SO(n+1)\big)/\text{Torsion}\\
&= \Lambda_{\mathbf{Z}}\big(\beta(\lambda_1), ..., \beta(\lambda_{n-3}), \varepsilon_{n+1}\big),
\end{aligned}$$

where $\beta : R(SO(n)) \longrightarrow \widetilde{K}^{-1}(SO(n))$ be the homomorphism of Abelian groups assigning to each representation $\rho : SO(n) \longrightarrow U(n+1)$ the homotopy class $\beta(\rho) = [i_n\rho] \in [SO(n), U] = \widetilde{K}^{-1}(SO(n))$, where $i_n : U(n+1) \to U$ is the canonical one, $U(n+1)$ and U be the $n-th$ and infinite unitary groups respectively and $\varepsilon_{n+1} \in K^{-1}(SO(n+1))$. We have, finally

$$K_*(C^*(S^n)) \cong \Lambda_{\mathbf{Z}}(\beta(\lambda_1), ..., \beta(\lambda_{n-3}), \varepsilon_{n+1}).$$

Moreover, the Chern character of $SU(n+1)$ was computed in [14], for all $n \geq 1$. Let us recall the result. Define a function

$$\phi : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \longrightarrow \mathbf{Z},$$

given by

$$\phi(n, k, q) = \sum_{i=1}^{k} (-1)^{i-1} \binom{n}{k-1} i^{q-1}.$$

Theorem 1.6. *Let $\mathbf{T}_n$ be a fixed maximal torus of $O(n)$ and $\mathbf{T}$ the fixed maximal torus of $SO(n)$, with Weyl groups $W := \mathcal{N}_{\mathbf{T}}/\mathbf{T}$, the Chern character of $C^*(S^n)$*

$$ch_{C^*} : K_*(C^*(S^n)) \longrightarrow HE_*(C^*(S^n))$$

is a isomorphism, given by

$$ch_{C^*}\big(\beta(\lambda_k)\big) = \sum_{i=1}^{n} ((-1)^{i-1} 2/(2i-1)!) \phi(2n+1, k, 2i) x_{2i+3}, \quad (k = 1, 2, ..., n-1)$$

$$ch_{C^*}(\varepsilon_{n+1}) = \sum_{i=1}^{n} \big((-1)^{i-1} 2/(2i-1)!)((1/2^n) \sum_{k=1}^{n} \phi(2n+1, k, 2i)\big) x_{2i+3}.$$

Proof. By Proposition 1.5, we have

$$K_*(C^*(S^n)) \cong K_*(\mathbb{C}(\mathcal{N}_{\mathbf{T}_n})) \cong K^*(\mathcal{N}_{\mathbf{T}_n})$$

and

$$HE_*(C^*(S^n)) \cong HE_*(\mathbb{C}(\mathcal{N}_{\mathbf{T}_n})) \cong H^*_{DR}(\mathcal{N}_{\mathbf{T}_n}) \qquad \text{(by Proposition 1.4).}$$

Now, consider the commutative diagram

$$\begin{tikzcd}
K_*\big(C^*(S^n)\big) \arrow[r, "ch_{C^*}"] \arrow[d, "\cong"] & HE_*\big(C^*(S^n)\big) \arrow[d, "\cong"] \\
K_*\big(\mathbb{C}(\mathcal{N}_{\mathbf{T}_n})\big) \arrow[r, "ch_{CQ}"] \arrow[d, "\cong"] & HE_*\big(\mathbb{C}(\mathcal{N}_{\mathbf{T}_n})\big) \arrow[d, "\cong"] \\
K^*\big(\mathcal{N}_{\mathbf{T}_n}\big) \arrow[r, "ch"] & H^*_{DR}\big(\mathcal{N}_{\mathbf{T}_n}\big)
\end{tikzcd}$$

Moreover, by the results of Watanabe [15], the Chern character $ch : K^*\big(\mathcal{N}_{\mathbf{T}_n}\big) \otimes \mathbb{C} \longrightarrow H^*_{DR}\big(\mathcal{N}_{\mathbf{T}_n}\big)$ is an isomorphism.

Thus, $ch_{C^*} : K_*\big(C^*(S^n)\big) \longrightarrow HE_*\big(C^*(S^n)\big)$ is an isomorphism (by Proposition 1.4 and 1.5), given by

$$ch_{C^*}(\beta(\lambda_k)) = \sum_{i=1}^{n} \big((-1)^{i-1}\, 2/(2i-1)!\big)\phi(2n+1,k,2i)x_{2i+3}, \ \ (k=1,2,...,n-1),$$

$$ch_{C^*}(\varepsilon_{n+1}) = \sum_{i=1}^{n} \big((-1)^{i-1} 2/(2i-1)!\big)\Big((1/2^n)\sum_{k=1}^{n}\phi(2n+1,k,2i)\Big)x_{2i+3},$$

where:

$$K_*\big(C^*(S^n)\big) \cong \Lambda_{\mathbf{Z}}(\beta(\lambda_1),...,\beta(\lambda_{n-3}),\varepsilon_{n+1}),$$
$$HE_*\big(C^*(S^n)\big) \cong \Lambda_{\mathbb{C}}(x_3,x_7,...,x_{2n+3}).$$

□

§2. Non-Commutative Chern character of $\mathbf{C}^*$-algebra of quantum spheres.

In this section, we at first recall definitions and main properties of compact quantum spheres and their representations. More precisely, for S^n, we define $C^*_\varepsilon(S^n)$, the C^*-algebras of compact quantum spheres as the C^*-completion of the *-algebra $\mathcal{F}_\varepsilon(S^n)$ with respect to the C^*-norm, where $\mathcal{F}_\varepsilon(S^n)$ is the quantized Hopf subalgebra of the Hopf algebra, dual to the quantized universal enveloping algebra $U(\mathcal{G})$, generated by matrix elements of the $U(\mathcal{G})$ modules of type **1** (see [3]). We prove that

$$C^*_\varepsilon(S^n) \cong \mathbb{C}(S^1) \oplus \bigoplus_{e\neq\omega\in W}\int_{S^1}^{\oplus}\mathcal{K}(H_{w,t})dt,$$

where $\mathcal{K}(H_{w,t})$ is the elementary algebra of compact operators in a separable infinite-dimensional Hilbert space $H_{w,t}$ and W is the Weyl group of S^n with respect to a maxinal torus $\mathbf{T}$.

After that, we first compute the K-groups $K_*\big(C^*_\varepsilon(S^n)\big)$ and the $HE_*(C^*_\varepsilon(S^n))$, respectively. Thereafter we define the Chern character of C^*-algebras quantum spheres, as a homomorphism from $K_*\big(C^*_\varepsilon(S^n)\big)$ to $HE_*\big(C^*_\varepsilon(S^n)\big)$, and we prove that $ch_{C^*} : K_*(C^*_\varepsilon(S^n)) \longrightarrow HE_*\big(C^*_\varepsilon(S^n)\big)$ is an isomorphism modulo torsions.

Let G be a complex algebraic group with Lie algebra $\mathcal{G}=$ Lie G and ε is real number, $\varepsilon \neq -1$.

Definition 2.1. ([3], Definition 13.1). *The quantized function algebra $\mathcal{F}_\varepsilon(G)$ is the subalgebra of the Hopf algebra dual to $U_\varepsilon(\mathcal{G})$, generated by the matrix elements of the finte-dimensional $U_\varepsilon(\mathcal{G})$-modules of type* **1**.

For compact quantum groups the unitary representation of $\mathcal{F}_\varepsilon(G)$ are parameterized by pairs (w,t), where t is an element of a fixed maximal torus of the compact real form of G and w is an element of the Weyl group W of $\mathbf{T}$ in G.

Let $\lambda \in P^+, V_\varepsilon(\lambda)$ be the irreducible $U_\varepsilon(\mathcal{G})$-module of type **1** with the highest weight λ. Then $V_\varepsilon(\lambda)$ admits a positive definite hermitian form (.,.), such that

$(xv_1, v_2) = (v_1, x^*v_2)$ for all $v_1, v_2 \in V_\varepsilon(\lambda), x \in U(\mathcal{G})$. Let $\{v^\nu_\mu\}$ be an orthogonal basis for weight space $V_\varepsilon(\lambda)_\mu$, $\mu \in P^+$. Then $\cup\{v^\nu_\mu\}$ is an orthogonal basis for $V_\varepsilon(\lambda)$. Let $C^\lambda_{\nu,s;\mu,r}(x) = \big(xv^r_\mu, v^s_\nu\big)$ be the associated matrix elements of $V_\varepsilon(\lambda)$. Then the matrix elements $C^\lambda_{\nu,s;\mu,r}$ (where λ runs throngh P^+, while (μ, r) and (ν, s) runs independently through the index set of a basis of $V_\varepsilon(\lambda)$ form a basis of $\mathcal{F}_\varepsilon(G)$ (see [3]).

Now very irreducible * - representation of $\mathcal{F}_\varepsilon(SL_2(\mathbb{C}))$ is equivalen to a representation belonging to one of the following two families, each of which is parameterized by $S^1 = \{t \in \mathbb{C}|\ |t| = 1\}$,

i) the family of one-dimensional representation $\mathcal{T}_t$

ii) the family π_t of representation in $\ell^2(\mathbb{N})$ (see [3])

Moreover, there exists a surjective homomorphism $\mathcal{F}_\varepsilon(G) \longrightarrow \mathcal{F}_\varepsilon(SL_2(\mathbb{C}))$ induced by the natural inclusion $SL_2(\mathbb{C}) \hookrightarrow G$ and by composing the representation π_{-1} of $\mathcal{F}(SL_2(\mathbb{C}))$ with this homomophis, we obtain a representation of $\mathcal{F}_\varepsilon(G)$ in $\ell^2(\mathbb{N})$ denoted by π_{s_i}, where s_i appears in the reduced decomposition $w = s_{i_1}.s_{i_2} \dots s_{i_k}$. More precisely, $\pi_{s_i} : \mathcal{F}_\varepsilon(G) \longrightarrow \mathcal{L}(\ell^2(\mathbb{N}))$ is of class CCR (see [11]), i.e its image is dense in the ideal of compact operators in $\mathcal{L}(\ell^2(\mathbb{N}))$.

The representation $\mathcal{T}_t$ is one-dimensional and is of the form

$$\mathcal{T}_t\big(C^\lambda_{\nu,s,\mu,r}\big) = \delta_{r,s}\delta_{\mu\nu} exp\,(2\pi\sqrt{-1}\mu(x)),$$

if $t = exp\,(2\pi\sqrt{-1}x) \in \mathbf{T}$, for $x \in$ Lie $\mathbf{T}$, (see [3]).

Proposition 2.2 ([3], 13.1.7). *Every irreducible unitary representation of $\mathcal{F}_\varepsilon(G)$ on a separable Hilbert space is the completion of a unitarizable highest weight representation. Moreover, two such representation are equivalent if and only if they have the same highest weight.* □

Proposition 2.3. ([3], 13.1.9). *Let $\omega = s_{i_1}.s_{i_2} \dots s_{i_k}$ be a resuced decomposition of an element w of the Weyl group W of G. Then*

*i) the Hilbert space tensor product $\rho_{w,t} = \pi_{s_{i_1}} \otimes \pi_{s_{i_2}} \otimes \cdots \otimes \pi_{s_{i_k}} \otimes \mathcal{T}_t$ is an irreducible *-representation of $\mathcal{F}_\varepsilon(G)$ which is associated to the Schubert cell S_w;*

ii) up to equivalence, the representation $\rho_{w,t}$ does not depend on the choice of the reduced decomposition of w;

*iii) every irreducible *-representation of $\mathcal{F}_\varepsilon(G)$ is equivalent to some $\rho_{w,t}$.* □

The sphere S^n, can be realized as the orbit under the action of the compact group $SU(n+1)$ of the highest weight vector v_0 in its natural $(n+1)$-dimensional representation $V^\hbar$ of $SU(n+1)$. If $t_{rs}, 0 \le r, s, \le n$, are the matrix entries of $V^\hbar$, the algebra of functions on the orbit is generated by the entries in the "first column" t_{s0} and their complex conjugates. In fact,

$$\mathcal{F}(S^n) := \mathbb{C}[t_{00}, \dots, t_{n0}, \bar{t}_{00}, \dots \bar{t}_{n0}]/\sim,$$

where "$\sim$" is the following equivalence ralation

$$t_{s0}\bar{t}_{s0} \Longleftrightarrow \sum_{s=0}^{n} t_{s0}.\bar{t}_{s0} = 1.$$

Proposition 2.4. ([3], 13.2.6). *The *-structure on Hopf algebra $\mathcal{F}_\varepsilon(SL_{n+1}(\mathbb{C}))$, is given by*

$$t^*_{rs} = (-\varepsilon)^{r-s}.q\det(\widehat{T}_{rs}),$$

where $\widehat{T}_{rs}$ is the matrix obtained by removing the r^{th} row and the s^{th} column from T.

Definition 2.5. ([3], 13.2.7). *The *-subalgebra of $\mathcal{F}_\varepsilon(SL_{n+1}(\mathbb{C}))$ generated by the elements t_{s0} and t^*_{s0}, for $s = 0,\dots n$, is called the quantized algebra of functions on the sphere S^n, and is denoted by $\mathcal{F}_\varepsilon(S^n)$. It is a quantum SL_{n+1} $(\mathbb{C})$-space.*

We set $z_s = t_{s0}$ from now on. Using Proposition 2.4, is is easy to see that the following relations hold in $\mathcal{F}_\varepsilon(S^n)$:

$$\begin{cases} z_r.z_s = \varepsilon^{-1}z_sz_r & \text{if} \quad r < s \\ z_r.z^*_s = \varepsilon^{-1}z^*_sz_r & \text{if} \quad r \neq s \\ z_r.z^*_r - z^*_r.z_r + (\varepsilon^{-2}+1)\sum\limits_{s>r} z_s.z^*_s = 0, & \\ \sum\limits_{s=0}^{n} z_s.z^*_s = 0. & \end{cases} \tag{*}$$

Hence, $\mathcal{F}_\varepsilon(S^n)$ has (*) as its defining relations. The construction of irreducible * -representations of $\mathcal{F}_\varepsilon(S^n)$, is given by

Theorem 2.6. ([3], 13.2.9). *Every irreducible *-representation of $\mathcal{F}_\varepsilon(S^n)$ is equivalent exactly to one of the following:*

i) the one-dimensional representation $\rho_{0,t}, t \in S^1$, given by $\rho_{0,t}(z^*_0) = t^{-1}, \rho_{0,t}(z^*_r) = 0$ if $r > 0$.

ii) the representation $\rho_{r,t}$, $1 \leq r \leq n$, $t \in S^1$, on the Hilbert space tensor product $\ell^2(\mathbb{N})^{\otimes r}$, given by

$$\rho_{r,t}(z^*_s)(e_{k_1}\otimes\cdots\otimes e_{k_r}) =$$

$$\begin{cases} \varepsilon^{-(k_1+\cdots k_s+s)}(1-\varepsilon^{-2(k_{s+1}+1)})^{1/2}e_{k_1}\otimes\cdots\otimes e_{k_s}\otimes e_{ks+1}+1\otimes e_{ks+2} & \\ \qquad\qquad\otimes\cdots\otimes e_{ks} & \text{if} \;\; s<r \\ t^{-1}.\varepsilon^{-(k_1+\cdots+k_r+r)}e_{k_1}\otimes\cdots e_{k_r} & \text{if} \;\; r=s \\ 0 & \text{if} \;\; s>r \end{cases}$$

The representation $\rho_{0,t}$ is equivalent to the restriction of the representation $\mathcal{T}_t$ of $\mathcal{F}_\varepsilon(SL_{n+1}(\mathbb{C}))$ (cf. 2.3); and for $r > 0, \rho_{r,t}$ is equivalent to the restriction of $\pi_{s_1}\otimes\cdots\otimes\pi_{s_r}\otimes\mathcal{T}_t$. □

From Theorem 2.6, we have

$$\bigcap_{(w,t)\in W\times\mathbf{T}} \ker\rho_{w,t} = \{e\},$$

i.e. the representation $\bigoplus\limits_{w\in W}\int_T^\oplus \rho_{w,t}dt$ is faithful and

$$\dim\rho_{w,t} = \begin{cases} 1 & \text{if } w = e \\ 0 & \text{if } w \neq e. \end{cases}$$

We recall now the definition of compact quantum of spheres C^*-algebra.

Definition 2.7. *The C^*-algebraic compact quantum sphere $C^*_\varepsilon(S^n)$ is the C^*-completion of the *-algebra $\mathcal{F}_\varepsilon(S^n)$ with respect to the C^*-norm*

$$\|f\| = \sup_\rho \|\rho(f)\| \quad (f \in \mathcal{F}_\varepsilon(S^n)),$$

*where ρ runs through the *-representations of $\mathcal{F}_\varepsilon(S^n)$ (cf., Theorem 2.6) and the norm on the right-hand side is the operator norm.*

It suffcies to show that $\|f\|$ is finite for all $f \in \mathcal{F}_\varepsilon(S^n)$, for it is clear that $\| \,.\, \|$ is a C^*-norm, i.e. $\|f.f^*\| = \|f\|^2$. We now prove the following result about the structure of compact quantum C^*-algebra of sphere S^n.

Theorem 2.8. *With notation as above, we have*

$$C^*_\varepsilon \cong \mathbb{C}(S^1) \oplus \bigoplus_{e\neq w\in W} \int_{S^1}^{\oplus} \mathcal{K}(H_{w,t})dt,$$

where $\mathbb{C}(S^1)$ is the algebra of complex valued continuous functions on S^1 and $\mathcal{K}(H)$ the ideal of compact operators in a separable Hilbert space H.

Proof: Let $w = s_{i_1}.s_{i_2}\dots s_{i_k}$ be a reduced decomposition of the element $w \in W$ into a product of reflections. Then by Proposition 2.6, for $r > 0$, the representation $\rho_{w,t}$ is equivalent to the restriction of $\pi_{s_{i_1}} \otimes \pi_{s_{i_2}} \otimes \cdots \otimes \pi_{s_{i_k}} \otimes \mathcal{T}_t$, where π_{s_i} is the composition of the homomorphism of $\mathcal{F}_\varepsilon(G)$ onto $\mathcal{F}_\varepsilon(SL_2(\mathbb{C}))$ and the representation π_{-1} of $\mathcal{F}_\varepsilon(SL_2(\mathbb{C}))$ in the Hilbert space $\ell^2(\mathbb{N})^{\otimes r}$; and the family of one-dimensional representations $\mathcal{T}_t$, given by

$$\mathcal{T}_t(a) = t,\;\; \mathcal{T}_t(b) = \mathcal{T}_t(c) = 0,\;\; \mathcal{T}_t(d) = t^{-1},$$

where $t \in S^1$ and a, b, c, d are given by: Algebra $\mathcal{F}_\varepsilon(SL_2(C))$ is generated by the matrix elements of type $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Hence, by construction, the representation $\rho_{w,t} = \pi_{s_{i_1}} \otimes \pi_{s_{i_2}} \otimes \cdots \otimes \pi_{s_{i_k}} \otimes \mathcal{T}_t$. Thus, we have

$$\pi_{si} : C^*_\varepsilon(S^n) \longrightarrow C^*_\varepsilon(SL_2(\mathbb{C})) \xrightarrow{\pi_{-1}} \mathcal{L}(\ell^2(\mathbb{N})^{\otimes r}).$$

Now, π_{s_i} is CCR (see, [11]) and so, we have $\pi_{s_i}\big(C^*_\varepsilon(S^n)\big) \cong \mathcal{K}(H_{w,t})$. Moreover $\mathcal{T}_t(C^*_\varepsilon(S^n)) \cong \mathbb{C}$.

Hence,

$$\begin{aligned}\rho_{w,t}\big(C^*_\varepsilon(S^n)\big) &= \big(\pi_{s_{i_1}} \otimes \cdots \otimes \pi_{s_{i_k}} \otimes \mathcal{T}_t\big)\big(C^*_\varepsilon(S^n)\big)\\ &= \pi_{s_{i_1}}\big(C^*_\varepsilon(S^n)\big) \otimes \cdots \otimes \pi_{s_{i_k}}\big(C^*_\varepsilon(S^n)\big) \otimes \mathcal{T}_t\big(C^*_\varepsilon(S^n)\big)\\ &\cong \mathcal{K}(H_{s_{i_1}}) \otimes \cdots \otimes \mathcal{K}(H_{s_{i_k}}) \otimes \mathbb{C}\\ &\cong \mathcal{K}(H_{w,t}),\end{aligned}$$

where $\quad H_{w,t} = H_{s_{i_1}} \otimes \cdots \otimes H_{s_{i_k}} \otimes \mathbb{C}.$

Thus, $\rho_{w,t}\big(C^*_\varepsilon(S^n)\big) \cong \mathcal{K}(H_{w,t})$.
Hence, $\bigoplus\limits_{w\in W} \int^\oplus_{S^1} \rho_{w,t}\big(C^*_\varepsilon(S^n)\big) \cong \bigoplus\limits_{w\in W} \int^\oplus_{S^1} \mathcal{K}(H_{w,t})dt$.

Now, recall a result of S. Sakai's from [11]: Let A be a commutative C^*-algebra and B be a C^*-algebra. Then, $C_0(\Omega, B) \cong A \otimes B$, where Ω is the spectrum space of A.

Applying this result, for $B = \mathcal{K}(H_{w,t}) \cong \mathcal{K}$ and $A = \mathbb{C}(W \times S^1)$ be a commutative C^*-algebra.
Thus, we have

$$C^*_\varepsilon(S^n) \cong \mathbb{C}(S^1) \oplus \bigoplus_{e\neq w\in W} \int^\oplus_{S^1} \mathcal{K}(H_{w,t})dt.$$

□

Now, we first compute the $K_*\big(C^*_\varepsilon(S^n)\big)$ and the $HE_*(C^*_\varepsilon(S^n))$ of C^*-algebra of quantum sphere S^n.

Propsition 2.9.

$$HE_*\big(C^*_\varepsilon(S^n)\big) \cong H^*_{DR}(W \times S^1).$$

Proof. We have

$$\begin{aligned}
HE_*\big(C^*_\varepsilon(S^n)\big) &= HE_*\big(\mathbb{C}(S^1) \oplus \bigoplus_{e\neq w\in W} \int^\oplus_{S^1} \mathcal{K}(H_{w,t})dt\big) \\
&\cong HE_*\big(\mathbb{C}(S^1)\big) \oplus HE_*\Big(\bigoplus_{e\neq w\in W} \int^\oplus_{S^1} \mathcal{K}(H_{w,t})dt\Big) \\
&\cong HE_*\big(\mathbb{C}(W \times S^1) \otimes \mathcal{K}\big) \qquad \text{(by Proposition 1.1 §1)} \\
&\cong HE_*\big(\mathbb{C}(W \times S^1)\big)
\end{aligned}$$

Since $C(W \times S^1)$ is a commutative C^*-algebra, by Proposition 1.5, §1, we have

$$HE_*\big(C^*_\varepsilon(S^n)\big) \cong HE_*\big(\mathbb{C}(W \times S^1)\big) \cong H^*_{DR}(W \times S^1)$$

Proposition 2.10.

$$K_*\big(C^*_\varepsilon(S^n)\big) \cong K^*(W \times S^1).$$

Proof. We have

$$\begin{aligned}
K_*\big(C^*_\varepsilon(S^n)\big) &= K_*\big(\mathbb{C}(S^1) \oplus \bigoplus_{e\neq w\in W} \int^\oplus_{S^1} \mathcal{K}(H_{w,t})dt\big) \\
&\cong K_*\big(\mathbb{C}(S^1)\big) \oplus K_*\Big(\bigoplus_{e\neq w\in W} \int^\oplus_{S^1} \mathcal{K}(H_{w,t})dt\Big) \\
&\cong K_*\big(\mathbb{C}(W \times S^1) \otimes \mathcal{K}\big) \\
&\cong K_*\big(\mathbb{C}(W \times S^1)\big) \qquad \text{(by proposition 1.1 §1).}
\end{aligned}$$

In virtue of Proposition 1.5, §1, we have

$$K_*\big(\mathbb{C}(W \times S^1)\big) \cong K^*(W \times S^1)$$

□

Theorem 2.11. *With notation above, the Chern character of C^*-algebra of quantum sphere $C^*_\varepsilon(S^n)$*

$$ch_{C^*} \; : \; K_*\big(C^*_\varepsilon(S^n)\big) \longrightarrow HE_*\big(C^*_\varepsilon(S^n)\big)$$

is an isomorphism.

Proof. By Proposition 2.9 and 2.10, we have:

$$\begin{aligned} HE_*\big(C^*_\varepsilon(S^n)\big) &\cong HE_*\big(\mathbb{C}(W \times S^1)\big) \cong H^*_{DR}(W \times S^1), \\ K_*\big(C^*_\varepsilon(S^n)\big) &\cong K_*\big(\mathbb{C}(W \times S^1)\big) \cong K^*(W \times S^1). \end{aligned}$$

Now, consider the commutative diagram

$$\begin{array}{ccc} K_*\big(C^*_\varepsilon(S^n)\big) & \xrightarrow{ch_{C^*_\varepsilon}} & HE_*\big(C^*_\varepsilon(S^n)\big) \\ \downarrow \cong & & \downarrow \cong \\ K_*\big(\mathbb{C}(W \times S^1)\big) & \xrightarrow{ch_{CQ}} & HE_*\big(\mathbb{C}(W \times S^1)\big) \\ \downarrow \cong & & \downarrow \cong \\ K^*\big(W \times S^1\big) & \xrightarrow{ch} & H^*_{DR}\big(W \times S^1\big) \end{array}$$

Moreover, following Watanabe [15], the $ch \; : \; K^*(W \times S^1) \otimes \mathbb{C} \longrightarrow H^*_{DR}(W \times S^1)$ is an isomorphism.

Thus, $ch_{C^*_\varepsilon} \; : \; K^*\big(C^*_\varepsilon(S^n)\big) \longrightarrow HE_*\big(C^*_\varepsilon(S^n)\big)$ in an isomorphism.

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Department of Mathematics, Vinh University, Vinh City, Vietnam